

KLEE-PHELPS CONVEX GROUPOIDS

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ABSTRACT. We prove that a pair of proximal Klee-Phelps convex groupoids $A(o), B(o)$ in a finite-dimensional normed linear space E are normed proximal, *i.e.*, $A(o) \delta B(o)$ if and only if the groupoids are normed proximal. In addition, we prove that the groupoid neighbourhood $N_z(o) \subseteq S_z(o)$ is convex in E if and only if $N_z(o) = S_z(o)$.

1. INTRODUCTION

Klee-Phelps groupoids are named after V.L. Klee and R.R. Phelps. It was Klee, who characterized a convex set in terms of a subset of a finite-dimensional Euclidean space that is the set of all points that are nearest a point in the space [3]. It was Phelps, who proved that for a subset S in a inner product space E , the set S_z of all points having $z \in S$ as the nearest point is a convex set [6]. Let S_z be the set of all points in E having $z \in S$ as the nearest point in S , defined by

$$S_z = \left\{ x \in E : \|x - z\| = \inf_{y \in S} \|x - y\| \right\}.$$

Lemma 1.1. (Phelps' Lemma [6]) *If E is an inner product space, $S \subset E$ and $z \in S$, then S_z is convex.*

The main result in this paper is that a pair of Klee-Phelps convex groupoids $A(o), B(o)$ in a finite-dimensional normed linear space are proximal if and only if $A(o), B(o)$ are normed proximal.

2. PRELIMINARIES

Let E be a finite-dimensional real normed linear space, $A, B \subset E$, $x, y \in E$. The Hausdorff distance $D(x, A)$ is defined by $D(x, A) = \inf \{\|x - y\| : y \in A\}$ and $\|x - y\|$ is the distance between vectors x and y . The Čech closure [7] of A (denoted by $\text{cl}A$) is defined by $\text{cl}A = \{x \in V : D(x, A) = 0\}$. The sets A and B are proximal (near) (denoted $A \delta B$), provided $\text{cl}A \cap \text{cl}B \neq \emptyset$ [2, 4]. A nonempty space endowed with a proximity relation is called a proximity space [5]. The space E endowed with the proximity relation δ is called a proximal linear space. The assumption made here is that each proximal linear space is a topological space that provides the structure needed to define proximity relations. The proximity relation δ defines a nearness relation between convex groupoids useful in many applications. A subset

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$K \subset E$ is *convex*, provided, for every pair points $x, y \in K$, the line segment \overline{xy} connecting x and y belongs to K .

Let $S \subset E, x, y \in S$, and S_x, S_y are nonempty Klee-Phelps nearest point sets. From Lemma 1.1, S_x, S_y are convex sets. Sets S_x, S_y are proximal if and only if $\|a - b\| = 0$ for some $a \in \text{cl}S_x, b \in \text{cl}S_y$. That is, convex sets S_x, S_y are near, provided the convex set S_x has at least one point in common with S_y . In effect, S_x, S_y are proximal if and only if $\text{cl}(S_x) \cap \text{cl}(S_y) \neq \emptyset$. A *convex hull* of a subset A in E is the smallest convex set that contains A .

Lemma 2.1. *If $\text{cl}(S_z)$ is a convex hull in E , then $S_z \subseteq \text{cl}S_z$.*

Proof. Let $x \in (X \setminus S_z)$ such that $x = y$ for some $y \in \text{cl}(S_z)$. Consequently, $x \in \text{cl}(S_z)$. Hence, $S_z \subseteq \text{cl}(S_z)$. \square

Let $S \subset E, z \in S, \varepsilon := \inf_{y \in S} \|x - y\|$. The neighborhood of a point in a Klee-Phelps convex set (briefly, N_z) is defined by

$$N_{z,\varepsilon} = \{x \in E : \|x - z\| \leq \varepsilon\}.$$

Lemma 2.2. *If S_z is a Klee-Phelps convex set in E , $S \subset E, z \in S$, then $N_{z,\varepsilon} \cap \text{cl}S_z \neq \emptyset$.*

Proof. For each $x \in S_z, \|x - z\| = \inf_{y \in S} \|x - y\|$. Consequently, $x \in N_{z,\varepsilon}$. Hence, $N_{z,\varepsilon} \cap \text{cl}S_z \neq \emptyset$. \square

Corollary 2.3. *If S_z is a Klee-Phelps convex set in E , $S \subset E, z \in S$, then $N_{z,\varepsilon} \delta S_z$.*

Proof. Immediate from Lemma 2.2 and the definition of proximity δ . \square

Theorem 2.4. *Let S_z be a Klee-Phelps convex set in E , $S \subset E, z \in S$. $N_{z,\varepsilon} \subseteq S_z$ is convex if and only if $N_{z,\varepsilon} = S_z$.*

Proof. $N_{z,\varepsilon} \subseteq S_z \Leftrightarrow \|x - z\| = \inf_{y \in S} \|x - y\|$ for each $x \in S_z \Leftrightarrow N_{z,\varepsilon} = S_z$. Hence, N_z is convex. \square

A groupoid is a system $S(\circ)$ that consists of a nonempty set S together with a binary operation \circ on S [1]. A Klee-Phelps groupoid is a system $S_z(\circ)$ that consists of a nonempty convex set $S_z \subset E$ together with a binary operation \circ on S_z such that $S_z(\circ) \subseteq S_z$.

Corollary 2.5. *Let $S_z(\circ)$ is a Klee-Phelps convex groupoid in E , $S \subset E, z \in S$. The neighbourhood groupoid $N_{z,\varepsilon}(\circ) \subseteq S_z(\circ)$ is convex if and only if $N_{z,\varepsilon}(\circ) = S_z(\circ)$.*

Proof. Immediate from Theorem 2.4. \square

Theorem 2.6. *Let U, V be proximal linear spaces, $z \in S \subset U, z' \in S' \subset V$ and let $S_z(\circ), S'_{z'}(\circ)$ be Klee-Phelps convex groupoids. Then*

$$\text{cl}(S_z) \cap \text{cl}(S'_{z'}) \neq \emptyset \Leftrightarrow S_z(\circ) \delta S'_{z'}(\circ).$$

Proof. Immediate from the definition of the proximity δ . \square

3. MAIN RESULTS

Let U^m and V^n be m - and n -dimensional proximal linear spaces, respectively, $m, n \in \mathbb{N}$. Also, let \mathcal{S}, T, s, t be the mappings given in

$$\begin{aligned} U^m \times U^m &\xrightarrow{\mathcal{S}} U^k, & U^m &\xrightarrow{s} U^k, \\ V^n \times V^n &\xrightarrow{T} V^k, & V^n &\xrightarrow{t} V^k. \end{aligned}$$

Lemma 3.1. *Let U^m, V^n be proximal linear spaces, $y, z \in S \subset U^m, y', z' \in S' \subset U^m, N_{y,\varepsilon}, S_z \subset U^m$ and $N_{y',\varepsilon}, S'_{z'} \subset V^n$.*

- 1° *If $m = n$ and $N_{y,\varepsilon} \delta N_{y',\varepsilon}$, then $S_z \delta S'_{z'}$.*
- 2° *If $m \neq n$ and $s(N_{y,\varepsilon}) \delta t(N_{y',\varepsilon})$, then $s(S_z) \delta t(S'_{z'})$.*
- 3° *If $m \neq n$ and $\mathcal{S}(N_{y,\varepsilon} \times N_{y',\varepsilon}) \delta T(N_{y',\varepsilon} \times N_{y',\varepsilon})$ then $\mathcal{S}(S_z \times S_z) \delta T(S'_{z'} \times S'_{z'})$.*

Proof.

- 1° Let $m = n$ and $N_{y,\varepsilon} \delta N_{y',\varepsilon}$. Then $\text{cl}(N_{y,\varepsilon}) \cap \text{cl}(N_{y',\varepsilon}) \neq \emptyset$. From Lemma 2.2, $N_{y,\varepsilon} \subseteq \text{cl}(S_z)$ and $N_{y',\varepsilon} \subseteq \text{cl}(S'_{z'})$. Consequently, $\text{cl}(S_z) \cap \text{cl}(S'_{z'}) \neq \emptyset$. Hence, $S_z \delta S'_{z'}$.
- 2° Let $m \neq n$ and $s(N_{y,\varepsilon}) \delta t(N_{y',\varepsilon})$. Then $\text{cl}(s(N_{y,\varepsilon})) \cap \text{cl}(t(N_{y',\varepsilon})) \neq \emptyset$. From Lemma 2.2, $N_{y,\varepsilon} \subseteq \text{cl}(S_z)$ and $N_{y',\varepsilon} \subseteq \text{cl}(S'_{z'})$. Consequently, $\text{cl}(s(S_z)) \cap \text{cl}(t(S'_{z'})) \neq \emptyset$. Hence, $s(S_z) \delta t(S'_{z'})$.
- 3° Let $m \neq n$ and $\mathcal{S}(N_{y,\varepsilon} \times N_{y',\varepsilon}) \delta T(N_{y',\varepsilon} \times N_{y',\varepsilon})$. Then $\text{cl}(\mathcal{S}(N_{y,\varepsilon} \times N_{y',\varepsilon})) \cap \text{cl}(T(N_{y',\varepsilon} \times N_{y',\varepsilon})) \neq \emptyset$. From Lemma 2.2, $N_{y,\varepsilon} \subseteq \text{cl}(S_z)$ and $N_{y',\varepsilon} \subseteq \text{cl}(S'_{z'})$. Consequently, $\text{cl}(\mathcal{S}(S_z \times S_z)) \cap \text{cl}(T(S'_{z'} \times S'_{z'})) \neq \emptyset$. Hence, $\mathcal{S}(S_z \times S_z) \delta T(S'_{z'} \times S'_{z'})$. □

Theorem 3.2. *Let U^m, V^n be proximal linear spaces, $N_{y,\varepsilon}(\circ) \subset U^m, N_{y',\varepsilon}(\circ) \subset V^n$ proximal neighborhood groupoids and let $S_z(\circ) \subset U^m, S'_{z'}(\circ) \subset V^n$ be proximal Klee-Phelps convex groupoids.*

- 1° *If $m = n$ and $N_{y,\varepsilon}(\circ) \delta N_{y',\varepsilon}(\circ)$ then $S_z(\circ) \delta S'_{z'}(\circ)$.*
- 2° *If $m \neq n$ and $s(N_{y,\varepsilon}(\circ)) \delta t(N_{y',\varepsilon}(\circ))$ then $s(S_z(\circ)) \delta t(S'_{z'}(\circ))$.*
- 3° *If $m \neq n$ and $\mathcal{S}(N_{y,\varepsilon}(\circ) \times N_{y',\varepsilon}(\circ)) \delta T(N_{y',\varepsilon}(\circ) \times N_{y',\varepsilon}(\circ))$ then $\mathcal{S}(S_z(\circ) \times S_z(\circ)) \delta T(S'_{z'}(\circ) \times S'_{z'}(\circ))$.*

Proof. Immediate from Lemma 3.1. □

Remark 3.3. Let $A, B \subset E, A \delta B$, provided $\|a - b\|$ for some $a \in \text{cl}A, b \in \text{cl}B$, i.e., $\text{cl}A \cap \text{cl}B \neq \emptyset$. From the definition of $N_{z,\varepsilon}$, a neighborhood of point $z \in E$, we know $\|x - z\| < \varepsilon$ for each $x \in E$ that is in $N_{z,\varepsilon}$. If a pair of neighborhoods $N_{z,\varepsilon}, N_{z',\varepsilon}$ are proximal, then the neighborhoods have at least one point x in common. For this reason, we can then write $\|x - z\| = \|x - z'\|$. This leads to what is known as normed proximity δ_P , i.e., $A \delta_P B$, provided $\|x - z\| = \|x - z'\|$ for some $x \in E, z \in A, z' \in B$. ■

Let (E, δ, δ_P) denote a finite-dimensional normed linear space endowed with proximities δ, δ_P (briefly, proximal linear space).

Lemma 3.4. *Let (E, δ, δ_P) be a proximal linear space, $A, B \subset E$.*

- (i) *If $A = N_{z,\varepsilon}, B = N_{z',\varepsilon}$, then $A \delta_P B \Rightarrow A \delta B$.*

(ii) If $A = S_z$, $B = S_{z'}$ then $A \delta_P B \Leftrightarrow A \delta B$.

Proof.

(i) $A = N_{z,\varepsilon} \subset E$, $B = N_{z',\varepsilon} \subset E$.

$$\begin{aligned} A \delta_P B &\Rightarrow \|x - z\| = \|x - z'\|, \text{ for some } x \in E, z \in A, z' \in B \\ &\Rightarrow x \in \text{cl}A \text{ and } x \in \text{cl}B \\ &\Rightarrow \text{cl}A \cap \text{cl}B \neq \emptyset \\ &\Rightarrow A \delta B. \end{aligned}$$

(ii) Let $A = S_z \subset E$ and $B = S_{z'} \subset E$.

$$\begin{aligned} A \delta B &\Leftrightarrow \text{cl}A \cap \text{cl}B \neq \emptyset \\ &\Leftrightarrow \exists x \in \text{cl}A \cap \text{cl}B \\ &\Leftrightarrow x \in \text{cl}A \text{ and } x \in \text{cl}B \\ &\Leftrightarrow \|x - z\| = \|x - z'\| \text{ for some } x \in E, z \in A, z' \in B \\ &\Leftrightarrow A \delta_P B. \end{aligned}$$

□

Theorem 3.5. Let (E, δ, δ_P) be a proximal linear space, $A, B, S, S' \subset E$, $z \in S, z' \in S'$.

(i) If $A(\circ) = N_{z,\varepsilon}(\circ)$, $B(\circ) = N_{z',\varepsilon}(\circ)$, then $A(\circ) \delta_P B(\circ) \Rightarrow A(\circ) \delta B(\circ)$.

(ii) If $A(\circ) = S_z(\circ)$, $B(\circ) = S_{z'}(\circ)$, then $A(\circ) \delta_P B(\circ) \Leftrightarrow A(\circ) \delta B(\circ)$.

Proof. Immediate from Lemma 3.4. □

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